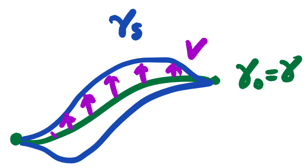
MATH 5061 Lecture on 4/8/2020

Recall: smooth curve $\gamma : [0,1] \to (M,g)$.

Energy of γ = $E(\gamma) := \int_0^1 \|\gamma'(t)\|_g^2 \, dt$ $\qquad L(\gamma) := \int_0^1 \|\gamma'(t)\|_g \, dt$

For 1-parameter family $\{\gamma_s\}_{s \in (-\varepsilon, \varepsilon)}$ w/. $\gamma_0 = \gamma$.



(1) $$\frac{d}{ds}\Big|_{s=0} E(\gamma_s) = \int_0^1 \langle V, D_{\gamma'}\gamma' \rangle \, dt + \langle V, \gamma' \rangle \Big|_{t=0}^{t=1}$$

$\Rightarrow$ γ crit. pt. to E $\Leftrightarrow$ $D_{\gamma'}\gamma' \equiv 0$ "geodesics".

(2) $$\frac{d^2}{ds^2}\Big|_{s=0} E(\gamma_s) = \int_0^1 \|D_{\frac{\partial}{\partial t}} V\|^2 - R(\gamma', V, \gamma', V) \, dt =: I(V,V)$$ index form

at a closed geodesic γ.

where $V := \frac{\partial}{\partial s}\Big|_{s=0} \gamma_s \in \mathcal{T}(\gamma^* TM)$

"variation field"

Def$^{\underline{n}}$: (Jacobi fields) $V \in \mathcal{T}(\gamma^* TM)$

$$\boxed{D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial t}} V + R(\gamma', V)\gamma' = 0}$$ Jacobi field eq$^{\underline{n}}$

Major Question: How does "curvature" affect "topology"?
(local, fine) (global, coarse)

E.g.) Gauss-Bonnet Thm: (Σ^2, g) cpt orientable surface w/o boundary

$$\int_\Sigma K \, da = 2\pi \chi(\Sigma) \qquad (\chi(\Sigma) := 2 - 2\text{genus})$$

Cor: $K > 0$ everywhere $\Rightarrow$ $\Sigma \approx S^2$

Q: What about in higher dimensions?

- which "curvature"? Riem, Ric, R?
- "topology" more complicated, π_1, π_k, H_*, H^* ……

Idea: sect. curv. $K > 0$ $\Rightarrow$ geodesics are tend to be "unstable".

Def^{n}: A geodesic γ is stable if $I(V,V) \geq 0$ $\forall V \in \Gamma(\gamma^* TM)$

(if γ has endpts, then we require $V = 0$ at the endpts.)

["stable" $\Leftrightarrow$ "infinitesimal local min."]

Synge Thm: Let (M^m, g) be cpt orientable Riem. manifold.

Suppose (i) $\dim M = m$ is even

(ii) M has positive sectional curvature

ie $K_p(\pi) > 0$ $\forall \pi^2 \subseteq T_pM$, $\forall p \in M$.

Then: M is simply-connected, i.e. $\pi_1(M) = 0$.

Remark: NOT true for odd dim. e.g. $M^3 = S^3/\Gamma$ Lens spaces

(like $\mathbb{RP}^3 = S^3/\{\pm 1\}$)

Proof: By Contradiction. Suppose $\pi_1(M) \neq 0$.

$\rightsquigarrow$ $\exists$ $\gamma_0 \subseteq M$ non-contractible closed loop.

Consider the "minimization" problem:

$$\inf_{\gamma \sim \gamma_0} E(\gamma) = E(\gamma_*) \quad \text{for some } \gamma_* \sim \gamma_0 .$$

Note: γ_* is a smooth, closed geodesic in (M,g).

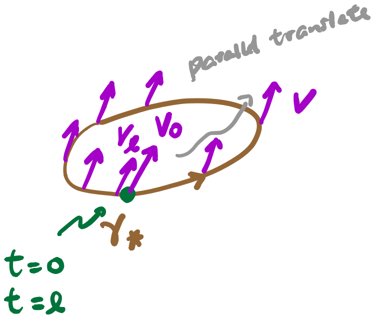
and γ_* is stable ($\because$ it is locally minimizing)

Claim: Any closed geodesic in (M,g) satisfying (i), (ii) are unstable.

Pf: Recall: $I_{\gamma_*}(V,V) := \int_0^{\ell} \| D_{\frac{\partial}{\partial t}} V \|_g^2 - R(\gamma_*', V, \gamma_*', V) \, dt$.

Goal: Find a parallel v.f. V, then $I(V,V) < 0$

$(D_{\frac{\partial}{\partial t}} V \equiv 0)$

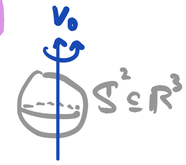


Find $V_0 \in T_{\gamma_*(0)}M$ s.t. it returns to the same vector after parallel transport along γ_* once.

let $P_{\gamma_*} : T_{\gamma_*(0)}M \longrightarrow T_{\gamma_*(\ell)}M$, $\gamma_*(\ell) = \gamma_*(0)$ — parallel transport is a linear isometry.

Note: $P_{\gamma_*}(\gamma_*'(0)) = \gamma_*'(0)$

So, $P_{\gamma_*} : \gamma_*'(0)^{\perp} \longrightarrow \gamma_*'(0)^{\perp} \in SO(m-1)$



Linear Alg. $\Rightarrow$ 1 is an eigenvalue of P_{γ_*} w/. some eigenvector $V_0 \neq 0$

Obtain a parallel v.f V by parallel transporting V_0.

$$\Rightarrow \quad I_{\gamma_*}(V,V) = \int_0^\ell \|\cancelto{0}{D_{\frac{\partial}{\partial t}} V}\|^2 - \underbrace{R(\gamma', V, \gamma', V)}_{K(\gamma'\wedge V)\,\|\gamma'\wedge V\|^2 \,>\, 0} \, dt \;<\; 0$$

$\because$ V parallel

$\blacksquare$

Bonnet-Myers Thm: Let (M^m, g) be a complete Riem. manifold.

Suppose: (i) $Ric \geq (m-1)g$ (ie. $Ric(X,X) \geq (m-1)\|X\|_g^2 \quad \forall X \in T_pM$)

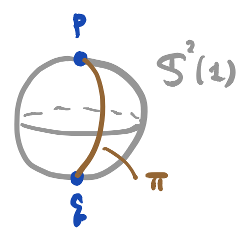
THEN: (a) $diam(M) := \sup\{\rho(p,q) \mid p,q \in M\} \leq \pi$

(b) M cpt

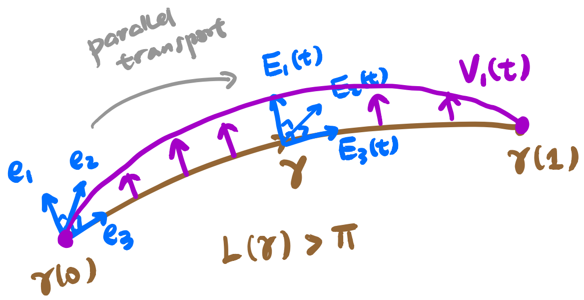
(c) $\pi_1(M)$ finite.

Example: $(M^m, g) = (S^m(1), g_{round})$, $Ric = (m-1)g$

$diam = \pi$, M cpt, $\pi_1 M = 0$.



"Proof": Claim: Any non-closed geodesic $\gamma : [0,1] \longrightarrow M$ s.t. $L(\gamma) > \pi$ is unstable. ($L(\gamma) = \ell$)



Take O.N.B. $\{e_1, \dots, e_m\}$ for $T_{\gamma(0)}M$. s.t. $e_m = \gamma'(0)/\ell$

parallel transport $\rightsquigarrow$ O.N.B. $E_1(t), \dots, E_m(t)$

parallel along γ

<u>Define</u>: $V_i(t) := \sin(\pi t)\, E_i(t)$, $i = 1, \dots, m-1$

<u>Note</u>: $D_{\frac{\partial}{\partial t}} V_i = \pi \cos(\pi t)\, E_i(t)$

$$\Rightarrow \quad I(V_i, V_i) = \int_0^1 \| D_{\frac{\partial}{\partial t}} V_i \|^2 - R(\gamma', V_i, \gamma', V_i)\, dt$$

$$= \int_0^1 \pi^2 \cos^2(\pi t) - \sin^2(\pi t)\, R(\gamma', E_i, \gamma', E_i)\, dt$$

$$= \int_0^1 \sin^2(\pi t) \left\{ \pi^2 - R(\gamma', E_i, \gamma', E_i) \right\} dt$$

Sum $i = 1, \dots, m-1$,

$$\sum_{i=1}^{m-1} I(V_i, V_i) = \int_0^1 \sin^2(\pi t) \left\{ (m-1)\pi^2 - \underbrace{Ric(\gamma', \gamma')}_{\substack{\text{w} \\ (m-1)\|\gamma'\|^2 = (m-1)\ell^2}} \right\} dt$$

$$\leq \int_0^1 \sin^2(\pi t) \cdot (m-1)(\pi^2 - \ell^2)\, dt < 0 \quad \text{if } \ell > \pi$$

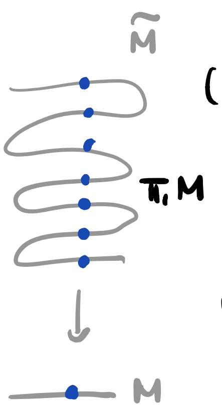
So, some $I(V_i, V_i) < 0 \Rightarrow \gamma$ is unstable.

Hopf-Rinow $\Rightarrow$ Any $p, q \in M$ is joined by a min. geodesic γ

Claim $\Rightarrow$ $L(\gamma) \leq \pi$ So, $\operatorname{diam}(M, g) \leq \pi$.

This also implies compactness .

Consider the universal cover $\widetilde{M}$ of M w/. pullback metric



$(\widetilde{M}, g)$ $Ric \geq (m-1)g$ complete $\Rightarrow$ $\operatorname{diam} \widetilde{M} \leq \pi$, $\widetilde{M}$ cpt

$\big\downarrow$

(M, g) $Ric \geq (m-1)g$ complete $\Rightarrow$ $\operatorname{diam} M \leq \pi$, M cpt

So, $\pi_1 M$ is finite.

$\square$

Q: What about negative curvatures? Think: $\mathbb{H}^m$ hyperbolic

Hadamard Thm: Let (M^m, g) be complete Riem. manifold.

Suppose: (i) $\pi_1(M) = 0$

(ii) non-positive sectional curvature (ie. $K \leq 0$)

THEN: $M^m \overset{\text{diffeo.}}{\approx} \mathbb{R}^m$. In fact, $\exp_p : \underset{\overset{\mathbb{R}^m}{\wr\wr}}{T_pM} \to M$ is a diffeomorphism.

Remarks: Dropping (i), then $\tilde{M} \overset{\text{diffeo.}}{\approx} \mathbb{R}^m$, so $\pi_k(M) = 0$ for all $k \geq 2$.

Q: What can we say about $\pi_1(M)$ when $K \leq 0$?

Preissman: $K < 0 \Rightarrow$ any non-trivial abelian $G \leq \pi_1(M)$ must be cyclic, ie $G \cong \mathbb{Z}$.

Yau: $K \leq 0 \Rightarrow$ ....

Submanifold theory

Consider $f : M^n \longrightarrow \bar{M}^{n+m}$ immersion.

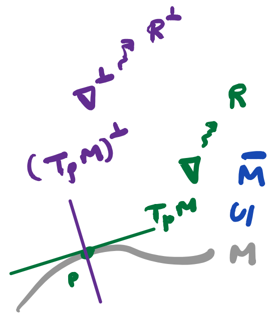
Def$^{\underline{n}}$: $f : (M^n, g) \longrightarrow (\bar{M}^{n+m}, \bar{g})$ is isometric immersion if $f^*\bar{g} = g$.

If $M^n \subset (\bar{M}^{n+m}, \bar{g})$, then the inclusion $\iota : (M^n, g) \to (\bar{M}, \bar{g})$ is an isometric embedding if $g = \bar{g}|_M$.

FACT: Locally, all isometric immersions arise this way (up to diffeo.)

Setup: $M^n \subset (\bar{M}^{n+m}, \bar{g})$ embedded submfd w. induced metric $g = \bar{g}|_M$

$(\bar{M}, \bar{g}) \rightsquigarrow \exists!$ Levi-Civita $\bar{\nabla} \rightsquigarrow \bar{R}$ Riem. curv.



From the orthogonal splitting:

$$T_p\bar{M} = T_pM \oplus (T_pM)^\perp$$

$$\rightsquigarrow \quad \bar{\nabla}_X s = \nabla_X s + \nabla^\perp_X s$$

Known: ∇ Levi-Civita on (M,g) on TM $\rightsquigarrow$ R Riem. curv. of (M,g)

and $\nabla^\perp$ metric-compatible connection of $(TM)^\perp$ $\rightsquigarrow$ $R^\perp$ "normal" curvature

Q: How are $\bar{R}$, R, $R^\perp$ related?

A: Gauss, Codazzi, Ricci equations. ("Constraint equations")

Recall: (2^{nd} f.f. & shape operator)

2^{nd} f.f : $\quad A(X,Y) := (\bar{\nabla}_X Y)^\perp \quad$ where $X, Y \in \Gamma(TM)$

shape operator : $\quad S_\eta(X) := -(\bar{\nabla}_X \eta)^T \quad$ where $X \in \Gamma(TM)$, $\eta \in \Gamma(T^\perp M)$

Remarks: (a) A is a $T^\perp M$-valued symm. $(0,2)$-tensor on M

$\Rightarrow \quad \vec{H} := \operatorname{tr} A \in \Gamma(T^\perp M)$ mean curvature vector of $M \subset \bar{M}$

[Def^n: $\vec{H} \equiv \vec{0} \iff M$ is a minimal submanifold.]

(b) S_η is a self-adjoint endomorphism of TM.

Fundamental eq^s for isometric immersions

Let $X, Y, Z, T \in \Gamma(TM)$, $\eta, \xi \in \Gamma(T^\perp M)$.

Gauss: $\langle \bar{R}(X,Y)Z, T\rangle = \langle R(X,Y)Z, T\rangle - \langle A(Y,T), A(X,Z)\rangle + \langle A(X,T), A(Y,Z)\rangle$

Ricci: $\langle \bar{R}(X,Y)\eta, \xi\rangle = \langle R^\perp(X,Y)\eta, \xi\rangle - \langle [S_\eta, S_\xi]X, Y\rangle$

Codazzi: $\langle \bar{R}(X,Y)Z, \eta\rangle = -(\bar{\nabla}_Y A)(X,Z,\eta) + (\bar{\nabla}_X A)(Y,Z,\eta)$.

where $A(X,Y,\eta) := \langle A(X,Y), \eta\rangle$.

Remark: Ricci eq^n always true in codim 1 case. ($\dim M = \dim \bar{M} - 1$)

Proof: **Recall:** $\bar{\nabla}_X Y = \nabla_X Y + A(X,Y)$ where $X, Y \in \Gamma(TM)$.

$$\bar{R}(X,Y)Z := \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z$$

$$= \bar{\nabla}_X \left(\nabla_Y Z + A(Y,Z) \right) - \bar{\nabla}_Y \left(\nabla_X Z + A(X,Z) \right)$$

$$- \nabla_{[X,Y]} Z - A([X,Y], Z)$$

$$= R(X,Y)Z + A(X, \nabla_Y Z) - A(Y, \nabla_X Z)$$

$$- S_{A(Y,Z)}(X) + \nabla_X^{\perp}(A(Y,Z))$$

$$+ S_{A(X,Z)}(Y) + \nabla_Y^{\perp}(A(X,Z)) - A([X,Y],Z)$$

tangential

normal

Take $\langle \cdot, T \rangle$,

$$\langle \bar{R}(X,Y)Z, T \rangle = \langle R(X,Y)Z, T \rangle - \langle S_{A(Y,Z)}(X), T \rangle$$

$$+ \langle S_{A(X,Z)}(Y), T \rangle$$

Note: $\langle S_{A(Y,Z)}(X), T \rangle = \langle -\bar{\nabla}_X (A(Y,Z)), T \rangle$

$$= \langle A(Y,Z), \bar{\nabla}_X T \rangle = \langle A(Y,Z), A(X,T) \rangle$$

Similarly for the term $\langle S_{A(X,Z)}(Y), T \rangle$, we obtain Gauss eq^{n}.

Take $\langle \cdot, \eta \rangle$.

$$\langle \bar{R}(X,Y)Z, \eta \rangle = \langle A(X, \nabla_Y Z), \eta \rangle - \langle A(Y, \nabla_X Z), \eta \rangle$$

$$+ \langle \nabla_X^{\perp}(A(Y,Z)), \eta \rangle - \langle \nabla_Y^{\perp}(A(X,Z)), \eta \rangle$$

$$- \langle A([X,Y], Z), \eta \rangle$$

work from R.H.S. of Codazzi eq^{n}:

$$(\bar{\nabla}_Y A)(X, Z, \eta) = Y\big(A(X,Z,\eta)\big) - A(\nabla_Y X, Z, \eta)$$
$$- A(X, \nabla_Y Z, \eta) - A(X, Z, \nabla^\perp_Y \eta)$$
$$= \langle \nabla^\perp_Y (A(X,Z)), \eta \rangle - \langle A(\nabla_Y X, Z), \eta \rangle$$
$$- \langle A(X, \nabla_Y Z), \eta \rangle$$

Similarly, for $(\bar{\nabla}_X A)(Y, Z, \eta)$. Combine to give Codazzi eq^{n}.

Finally, for Ricci eq^{n}, we consider

$$\bar{R}(X,Y)\eta := \bar{\nabla}_X \bar{\nabla}_Y \eta - \bar{\nabla}_Y \bar{\nabla}_X \eta - \bar{\nabla}_{[X,Y]}\eta$$
$$= \bar{\nabla}_X\big(-S_\eta(Y) + \nabla^\perp_Y \eta\big) - \bar{\nabla}_Y\big(-S_\eta(X) + \nabla^\perp_X \eta\big)$$
$$+ S_\eta([X,Y]) - \nabla^\perp_{[X,Y]}\eta$$
$$= \text{some tangential parts} \ldots$$
$$\left.\begin{aligned} &+ A(X, -S_\eta(Y)) - A(Y, -S_\eta(X)) \\ &+ R^\perp(X,Y)\eta \end{aligned}\right\} \text{normal parts}$$

Take $\langle \cdot, \xi \rangle$,

$$\langle \bar{R}(X,Y)\eta, \xi \rangle = \langle R^\perp(X,Y)\eta, \xi \rangle - \langle \bar{\nabla}_X (S_\eta(Y)), \xi \rangle$$
$$+ \langle \bar{\nabla}_Y (S_\eta(X)), \xi \rangle$$

On the other hand,

$$\langle \bar{\nabla}_Y (S_\eta(X)), \xi \rangle = -\langle S_\eta(X), \bar{\nabla}_Y \xi \rangle$$
$$= \langle S_\eta(X), S_\xi(Y) \rangle$$
$$= \langle S_\xi S_\eta(X), Y \rangle \Rightarrow \text{Ricci eq}^{n}$$

$\square$